SOME GENERALIZATIONS OF THE PROBLEM OF SPHERICAL VORTICES

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A generalization of Hill's solution [2], in which the velocity field is essentially three-dimensional with retention of cylindrical symmetry and the potentiality of the external streamline flow, is given in [1].

In the present report a vortex formation is constructed consisting of three-dimensional vortex flow in the gaps between concentric spheres of arbitrary radii and in the inner sphere for different dependences of the Bernoulli integral and the circulation on the stream function (according to [3], these functions depend only on the stream function).

Without a limitation of generality, let us consider vortex formations consisting of two flows: one in the gap between concentric spheres with radii a and a_0 ($a > a_0$) and the other in the inner sphere (the first flow will be called the outer vortex and the second, the inner vortex), moving with a constant velocity u in a fluid quiescent at infinity.

In spherical coordinates the dimensionless stream function $\psi_0(\rho, \vartheta)$ of flow by a potential stream over a sphere has the form [4]

$$\psi_0(\rho, \vartheta) = (1/2)\rho^2(1 - 1/\rho^3)\sin^2\vartheta.$$

Here and later all the quantities are reduced to dimensionless form (using the scale u, a).

Let us consider the vortex flow within concentric spheres with the circulation $\Gamma(\psi)$ and the Bernoulli integral $F(\psi)$ in the form

$$E(\psi) = -F(\psi), \; F(\psi) = A_i^0 + A_i\psi \leqslant 0, \qquad \Gamma(\psi) = k_i\psi, \; A_i^0, A_i, k_i = \text{const}. \tag{1}$$

The constant k_i is connected with A_i; this connection is discussed below.

With allowance for (1) the equations and boundary conditions for the stream function in a spherical coordinate system rigidly connected with the vortex formation have the form [3]

$$\begin{gathered}\frac{\partial^2\psi}{\partial\rho^2} + \frac{\sin\vartheta}{\rho^2}\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\vartheta}\frac{\partial\psi}{\partial\vartheta}\right) + k_i^2\psi + A_i\rho^2\sin^2\vartheta = 0,\\ v_\rho = \frac{1}{\rho^2\sin\vartheta}\frac{\partial\psi}{\partial\vartheta}, \quad v_\vartheta = -\frac{1}{\rho\sin\vartheta}\frac{\partial\psi}{\partial\rho}, \quad v_\varphi\rho\sin\vartheta = \Gamma(\psi),\\ \psi(\rho_i', \vartheta) = 0, \quad \psi(\rho_i, \vartheta) = 0,\\ \left.\frac{\partial\psi_i}{\partial\rho}\right|_{\rho=\rho_i} = \left.\frac{\partial\psi_{i-1}}{\partial\rho}\right|_{\rho=\rho_i} = g_{i-1}\sin^2\vartheta,\end{gathered} \tag{2}$$

where $\psi_{i-1}(\rho, \vartheta)$ is the stream function for the external streamline flow. The boundary conditions consist of the conditions of nonpenetration at the surfaces $\rho = \rho_i'$ and $\rho = \rho_i$ and equality of the axial velocity at the surface $\rho = \rho_i'$. Equality of the azimuthal and radial velocities is automatically satisfied. Hence, for the vortex flow in the gap between the spheres (i = 1) we have

$$\rho_1' = 1, \; \rho_1 = \rho_0 \left(\rho_0 = \frac{a_0}{a}\right), \quad g_0 = \frac{3}{2}.$$

The solution of the problem (2) is sought in the form $\psi(\rho, \vartheta) = f(\rho)\sin^2\vartheta$, and then the expression for the stream function of the outer vortex has the form

$$\psi(\rho, \vartheta) = \frac{A_1\sqrt{\rho}}{k_1^2}\left\{\frac{\left[\rho_0^{3/2}J_{3/2}(k_1) - J_{3/2}(k_1\rho_0)\right]}{J_{-3/2}(k_1\rho_0)J_{-3/2}(k_1) - J_{-3/2}(k_1)J_{3/2}(k_1\rho_0)}\left[J_{-3/2}(k_1\rho) - \frac{J_{-3/2}(k_1)J_{3/2}(k_1\rho)}{J_{3/2}(k_1)}\right] + \frac{J_{3/2}(k_1\rho)}{J_{3/2}(k_1)} - \rho^{3/2}\right\}\sin^2\vartheta,$$

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$$A_1 = g_0 k_1^2 \left\{ \frac{[\rho_0^{3/2} J_{3/2}(k_1) - J_{3/2}(k_1\rho_0)]}{J_{-3/2}(k_1\rho_0) J_{3/2}(k_1) - J_{-3/2}(k_1) J_{3/2}(k_1\rho_0)} \times \right.$$

$$\times \left\{ \left[k_1 J_{-5/2}(k_1) + 2J_{-3/2}(k_1) - \frac{J_{-3/2}(k_1)}{J_{3/2}(k_1)} [k_1 J_{1/2}(k_1) - J_{3/2}(k_1)] \right] \right\} + \left. \frac{k_1 J_{1/2}(k_1) - \frac{3}{2} J_{3/2}(k_1)}{J_{3/2}(k_1)} - 2 \right\}^{-1}. \quad (3)$$

Let us consider the flow in the inner sphere (i = 2); then

$$\rho_2' = \rho_0, \quad \rho_2 = 0,$$

$$g_1 = \frac{A_1}{k_1^2} \left\{ \frac{\rho_0^{3/2} J_{3/2}(k_1) - J_{3/2}(k_1\rho_0)}{[J_{-3/2}(k_1\rho_0) J_{3/2}(k_1) - J_{-3/2}(k_1) J_{3/2}(k_1\rho_0)] \sqrt{\rho}} \times \right.$$

$$\times \left\{ [k_1\rho_0 J_{-3/2}(k_1\rho_0) + 2J_{-3/2}(k_1\rho_0)] - \frac{J_{-3/2}(k_1)}{J_{3/2}(k_1)} [k_1\rho_0 J_{1/2}(k_1\rho_0) - \right.$$

$$\left. - J_{3/2}(k_1\rho_0)] + \frac{k_1\rho_0 J_{1/2}(k_1\rho_0) - J_{3/2}(k_1\rho_0)}{\sqrt{\rho_0} J_{3/2}(k_1)} - 2\rho_0 \right\} = \frac{A_1}{k_1^2} f(k_1),$$

$$\psi(\rho, \vartheta) = \frac{A_2 \sqrt{\rho}}{k_2^2} \left[\rho_0^{3/2} \frac{J_{3/2}(k_2\rho)}{J_{3/2}(k_2)} - \rho^{3/2} \right] \sin^2\vartheta, \quad (4)$$

$$A_2 = \left\{ \frac{\rho_0^{3/2} [k_2\rho_0 J_{1/2}(k_2\rho_0) - J_{3/2}(k_2\rho_0)]}{\sqrt{\rho_0} J_{3/2}(k_2\rho_0)} - 2\rho_0 \right\} k_2^2 g_1 = \frac{k_2^2 A_1}{k_1^2} f(k_1) g(k_2).$$

From Eqs. (4) it is seen that the same functions $\Gamma(\psi)$ and $F(\psi)$ in the two vortices are possible only when $f(k)g(k) = 1$. As $\rho_0 \to 0$, Eqs. (3) change into the expressions obtained in [1].

A numerical calculation of the flow within the vortex formation under consideration was carried out on the basis of (3) and (4).

The streamlines for equally spaced values of ψ are plotted in Fig. 1 for the case of $k_1 = k_2 = 1$ and $\rho_0 = 0.5$. The largest absolute values of the stream function in the inner and outer vortices are reached at the points $(0.25, \pi/2)$ and $(0.62, \pi/2)$, respectively. These are not critical points, since the azimuthal velocity components are different from zero.

The distribution of the azimuthal velocity component is given in Fig. 2. The largest absolute value of v_φ is reached at the points $(0.05, \pi/2)$ and $(0.62, \pi/2)$ for the inner and outer vortices, respectively:

$$|v_{\varphi\max}| = 1.69 \text{ and } |v_{\varphi\max}| = 0.3.$$

Taking $A_2 \to 0$ in Eqs. (4), we obtain

$$\psi(\rho, \vartheta) = \frac{g_1 \sqrt{\rho} J_{3/2}(k_2\rho) \sqrt{\rho_0} \sin^2\vartheta}{[k_2\rho_0 J_{1/2}(k_2\rho_0) - J_{3/2}(k_2\rho_0)]}, \quad (5)$$

where k_2 are the positive roots of the equation $J_{3/2}(k_2\rho_0) = 0$. Thus, a vortex formation is constructed in which the flow in the outer vortex is described by Eq. (3), while in the inner vortex there is uniform helical flow. If $k_2 = \nu_m$ is the m-th positive root of the equation $J_{3/2}(k_2\rho_0) = 0$, then the inner vortex divides into m isolated vortices which lie in the gaps between concentric spheres with radii $\rho_i = \nu_i/\nu_m$, while the functions $\Gamma(\psi)$ and $F(\psi)$ are the same in all the vortices. From Eq. (3) it is seen that the flow in the outer vortex is uniform helical flow when $A_1 = 0$. Then the stream function has the form

$$\psi(\rho, \vartheta) = \left[-\frac{J_{-3/2}(k_1)}{J_{3/2}(k_1)} J_{3/2}(k_1\rho) + J_{-3/2}(k_1\rho) \right] g_0 \times$$

$$\times \left\{ 2J_{-3/2}(k_1) + k_1 J_{-5/2}(k_1) - \frac{J_{-3/2}(k_1)}{J_{3/2}(k_1)} [k_1 J_{1/2}(k_1) - J_{3/2}(k_1)] \right\}^{-1} \sqrt{\rho} \sin^2\vartheta, \quad (6)$$

where k_1 is the m-th positive root of the transcendental equation

$$J_{+3/2}(k_1) J_{-3/2}(k_1\rho_0) - J_{-3/2}(k_1) J_{3/2}(k_1\rho_0) = 0.$$

Now if g_1 in Eq. (5) is replaced by

$$g_2 = \left\{ 2J_{3/2}(k_1\rho_0) + k_1\rho_0 J_{-5/2}(k_1\rho_0) - \frac{J_{-3/2}(k_1)}{J_{3/2}(k_1)} [k_1\rho_0 J_{1/2}(k_1\rho_0) - J_{3/2}(k_1\rho_0)] \right\} \rho_0^{-1/2} \left\{ 2J_{-3/2}(k_1) + \right.$$

$$\left. + k_1 J_{-5/2}(k_1) + [J_{3/2}(k_1) - k_1 J_{1/2}(k_1)] \frac{J_{-3/2}(k_1)}{J_{3/2}(k_1)} \right\}^{-1} g_0,$$

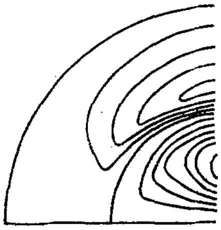

Fig. 1

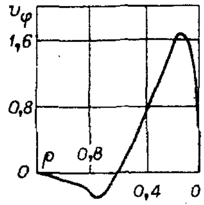

Fig. 2

then we construct a vortex formation for which there is uniform helical flow described by Eqs. (6) and the modified Eq. (5) in the outer and inner vortices.

The case of $k_i = 0$ corresponds to two-dimensional flow in the vortices, i.e., the azimuthal velocity component is absent. We write the solutions of the problem (3) with $k_i = 0$ for the inner and outer vortices, respectively, as

$$\psi(\rho, \vartheta) = \frac{g_1}{2\rho_0^3}[\rho^4 - \rho_0^2\rho^2]\sin^2\vartheta; \tag{7}$$

$$\psi(\rho, \vartheta) = \frac{g_0 \sin^2\vartheta}{5\rho_0^3 - 3\rho_0^5 - 2}\left[(1 - \rho_0^5)\rho^2 + \frac{\rho_0^3(\rho_0^2 - 1)}{\rho} + (\rho_0^3 - 1)\rho^4\right]. \tag{8}$$

If we replace g_1 by g_2 or by

$$g_3 = \frac{g_0}{2\rho_0^3 - 3\rho_0^5 - 2}[2(1 - \rho_0^5)\rho_0 - (\rho_0^2 - 1)\rho_0 + (\rho_0^3 \div 1)4\rho_0^3]$$

in Eq. (7), then the outer vortex is assigned by Eq. (6) or (8) while the inner vortex is assigned by the modified Eq. (7). Replacing g_1 by g_3 in Eqs. (4), we obtain two-dimensional vortex flow in the outer vortex and three-dimensional vortex flow or uniform helical flow, respectively, in the inner vortex.

The problem of the potential flow over a vortex formation consisting of m vortices lying in the gaps between concentric spheres of arbitrary radii with distribution functions in the form (1) in each vortex can be solved analogously. If the functions are assigned in the form (1), then one can be confined to the requirement that the stream functions at the surfaces of the concentric spheres be constants, with the exception of the outer sphere, and then the azimuthal velocities at these surfaces will differ from zero and $k_i = k$.

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